On the Validity of Macroscopic Models

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The validity of macroscopic models in the limit of large mean-free path is examined by solving a one-dimensional model equation, where the term $v \nabla f_1$ is retained in the kinetic equation. A standard thirteen-moments approximation yields accurate results in all collisionality regimes if the fluid velocity is sufficiently small. In contrast, the Chapman-Enskog scheme is only accurate in the collision-dominated regime.

1. Introduction

The gross macroscopic properties of laboratory plasmas relating to equilibrium, stability and transport are of special interest. Tokamak transport and confinement are major concerns of theory. A new approach taking flows consistently into account was recently presented by Weitzner and Kerner [1]. In this model the Grad-Lüst-Schlüter-Shafranov equilibrium relation for tokamaks

$$\Delta^* \psi = -4\pi (r^2 p'(\psi) + I(\psi) I'(\psi)),$$

relating the poloidal flux ψ, the pressure $p = p(\psi)$ and the poloidal current profile $I = I(\psi)$, is generalized to include a pressure tensor as well as small but finite mass flow. The density, pressure and flow profiles are then no longer surface quantities but vary on a flux surface poloidally up to the order of the inverse aspect ratio. This anisotropy is sustained by transport which can be compatible with the observed magnitude. Naturally there exist numerous extensions of the standard ideal MHD theory, such as the Braginskii model [2], the moment approaches of Grad [3] and Schlüter [4], the double-adiabatic theory [5] or the neoclassical MHD model of Callen et al. [6].

Since relevant tokamak devices operate in regimes with large mean-free path, the validity of macroscopic models is discussed here, particularly with respect to collisionality. It is recalled that the fluid description yields a set of equations in configuration space for the velocity moments of increasing order obtained from the appropriate kinetic equation. This infinite chain of equations is terminated by introducing specific closure schemes. Since the Larmor radius is small compared to the plasma radius, the validity of macroscopic models need only be discussed in the direction parallel to the magnetic field. When the mean-free path length l_f is smaller than the connection length l_c, the collision-dominated fluid regime applies. However, we shall prove the extension of fluid models for large collision times. Assuming that the distribution function is close to a local Maxwellian, we apply Grad's moment approach [3]. By solving a one-dimensional model equation in the parallel direction exactly we can discuss the accuracy of Grad's moment method, which for the one-dimensional case coincides with Schlüter's method, and compare it with the usual Chapman-Enskog scheme. It is noted that the effect of trapped particles is not included in this model.

2. Model Equation

Following the standard derivation of macroscopic models we begin with the Boltzmann equation for the single-particle distribution function $f = f(\mathbf{r}, \mathbf{v}, t)$:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{F} \cdot \nabla_v f = \left(\frac{\partial f}{\partial t}\right)_c, \qquad (1)$$

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where $m \cdot \mathbf{F}$ is the force and the right-hand side is the collision operator. If the collision operator is simplified to $-(f - f_0)/\tau$, where τ is the average collision time, the kinetic equation assumes for $f = f_0 + f_1$ the form of the Krook model

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{F} \cdot \nabla_v f = - f_1/\tau. \tag{2}$$

In the Chapman-Enskog expansion the terms containing f_1 on the left-hand side are dropped. However, the term $\mathbf{v} \cdot \nabla$ can become as large as $1/\tau$ when the scale length of the variation becomes comparable to the mean-free path (Lackner [7]). It is an essential point of Grad's and Schlüter's moment approach that this term is automatically included. It is noted that the collision operator in (2) does not conserve mass, momentum and energy. This is easily improved, however, by adding appropriate constants. For example, the form

$$\begin{aligned}\left(\frac{\partial f}{\partial t}\right)_c &= - f/\tau + n(\mathbf{r}, t) \cdot f_0(\mathbf{v})/\tau \\ &= - f_1/\tau + \frac{n-1}{\tau} \cdot f_0,\end{aligned} \tag{3}$$

where $n(\mathbf{r}, t) = \int_{-\infty}^{+\infty} f \, \mathrm{d}^3 v$ and $1 = \int_{-\infty}^{+\infty} f_0 \, \mathrm{d}^3 v$, trivially conserves mass. In [8] a representation in which mass, momentum and energy is conserved, is given. Since the objective of the paper is to study the validity of different fluid models with respect to collisionality, but not specific applications, it is sufficient to treat the collision operator in the simple form of (2).

On the assumption of steady state and zero-force term the problem assumes the form

$$\mathbf{v} \cdot \nabla (f_0 + f_1) = - f_1/\tau. \tag{4}$$

It is emphasized that this equation still contains the critical term $\mathbf{v} \cdot \nabla f_1$ and therefore allows to discuss the validity of the moment approach with respect to collisionality. Further simplifications are made in order to make the model tractable for complete analytical solution. The problem is therefore reduced to one dimension. Then f_0 is given by a local Maxwellian with inhomogeneous flow $u(x)$:

$$f_0 = 1/\sqrt{2\pi v_{\text{th}}^2} \exp\left\{-1/2 \left(\frac{v - u(x)}{v_{\text{th}}}\right)^2\right\}. \tag{5}$$

The problem may be further simplified by introducing a typical wave vector k for the variation of f_1 with respect to x; i.e. by replacing ∇f_1 by $i k f_1$, which yields the following model equation:

$$(i k \tau v + 1) f_1 = - \tau v \frac{\partial f_0}{\partial x}. \tag{6}$$

The velocities from here on are normalized to the thermal velocity and the length x to the scale length k^{-1}. Then the collision time τ is replaced by γ, and in these units the model equation assumes the form

$$(i \gamma v + 1) f_1 = - \gamma v (v - u) \frac{\partial u}{\partial x} f_0. \tag{6b}$$

The dimensionless parameter $\gamma = k \tau v_{\text{th}}$ is the ratio of l_f to the typical plasma length. For $\gamma = 1$ the mean-free path l_f is equal to the scale length k^{-1} which in a tokamak would be given by the connection length l_c. It is assumed that the confinement time is long enough to allow some (not necessarily very many!) collisions for relaxation towards a Maxwellian distribution function. The velocity u is restricted in the collisionless limit such that $\gamma \cdot u$ is bounded, i.e. $\lim_{\gamma \to \infty} |\gamma \cdot u| \le K < \infty$. This implies that the distance $u(\tau \cdot v_{\text{th}})$ has to be smaller than the product of K and the scale length k^{-1}.

The exact solution of (6) is obvious. Grad's and Schlüter's moment methods consist in expanding f_1 in Hermite polynomials with the argument at the shifted velocity:

$$\begin{aligned} f_1 &= - f_0 \frac{\partial u}{\partial x} \frac{\gamma v (v - u)}{1 + i \gamma v} \\ &= f_0 \frac{\partial u}{\partial x} i \sum_{n=0}^{\infty} g_n \frac{1}{n!} \mathrm{He}_n(v - u(x)). \end{aligned} \tag{7}$$

The polynomials $\mathrm{He}_n(x)$ are associated with the weight function $\tilde{w}(x) = \exp(-x^2/2)$ and obey the orthogonalization relation

$$\int_{-\infty}^{+\infty} e^{-x^2/2} \, \mathrm{He}_n(x) \cdot \mathrm{He}_m(x) \, \mathrm{d}x = \delta_{nm} \, n! \sqrt{2\pi}, \tag{8}$$

with the Kronecker symbol δ_{nm}. The Rodriguez formula yields the following representation for the Hermite polynomials:

$$\mathrm{He}_n(x) = (-1)^n e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2/2}. \tag{8}$$

Since the integral $\int_{-\infty}^{+\infty} \left| \frac{v}{1 + i \gamma v} (v - u) \frac{\partial u}{\partial x} \right|^2 f_0 \, \mathrm{d}v$ exists, the expansion (7) for f_1 converges, see e.g. Batemann chapter 10.19, [9].

In the following the exact solution is derived together with an approximate one obtained by applying truncation in the expansion (7) and corresponding equations for the coefficients g_n. This allows one to discuss the accuracy of approximative solutions such as occur in more realistic problems. The underlying formulae can be found in [10–11].

i) Exact Solution

Multiplying (7) by $\mathrm{He}_n(v-u)$ and integrating with respect to the velocity yields for the coefficients g_n:

$$g_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}v \frac{i\gamma v}{1+i\gamma v} w\, e^{-w^2/2}\, \mathrm{He}_n(w),$$

where $w(x) = v - u(x)$.
If the quantity

$$\xi = i/\gamma - u(x) \quad \text{with} \quad \mathrm{Im}(\xi) > 0 \tag{10}$$

is introduced, it follows that

$$\frac{i\gamma}{1+i\gamma v} = \frac{1}{w-\xi}$$

and further

$$\frac{v\cdot w}{w-\xi} = w + i/\gamma + (i/\gamma)\,\xi\,\frac{1}{w-\xi}.$$

Then g_n is given by

$$g_n = (i/\gamma)\,\delta_{n0} + \delta_{n1} + \frac{i\xi}{\gamma}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}w\, \mathrm{He}_n(w) \frac{1}{w-\xi} e^{-w^2/2}.$$

By inserting the Rodriguez formula one obtains

$$g_n = (i/\gamma)\,\delta_{n0} + \delta_{n1} + \frac{i\xi}{\gamma}\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} \mathrm{d}w\,(-1)^n \frac{1}{w-\xi}\frac{\mathrm{d}^n}{\mathrm{d}w^n} e^{-w^2/2}. \tag{11}$$

Integration by parts yields with

$$\frac{\mathrm{d}}{\mathrm{d}w}\frac{1}{w-\xi} = -\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{1}{w-\xi}$$

$$g_n = (i/\gamma)\,\delta_{n0} + \delta_{n1} + \frac{i\xi}{\gamma}\frac{1}{\sqrt{2\pi}}(-1)^n \cdot \frac{\mathrm{d}^n}{\mathrm{d}\xi^n} \int_{-\infty}^{+\infty} \mathrm{d}w \frac{1}{w-\xi} e^{-w^2/2}. \tag{12}$$

With $z = \xi/\sqrt{2}$ we define

$$i\pi\, y(z) = \int_{-\infty}^{+\infty} \mathrm{d}w \frac{1}{w-\xi} e^{-w^2/2}. \tag{13}$$

This function is related to the complementary error function

$$\mathrm{erfc}(z) = 2/\sqrt{\pi} \int_{z}^{+\infty} \mathrm{d}t\, e^{-t^2}. \tag{14}$$

When for ξ with $\mathrm{Im}(\xi) > 0$ the relation

$$\frac{1}{i(w-\xi)} = \int_{0}^{+\infty} \mathrm{d}s\, e^{-i(w-\xi)s}$$

is inserted in (13) we obtain

$$i\pi\, y(z) = i \int_{-\infty}^{+\infty} \mathrm{d}w \int_{0}^{+\infty} \mathrm{d}s\, e^{-w^2/2 - i(w-\xi)s} = i\sqrt{2\pi} \int_{0}^{+\infty} \mathrm{d}s\, e^{-s^2/2 + i\xi s}. \tag{13b}$$

Then the following representation for $y(z)$ is derived:

$$y(z) = e^{-z^2}\cdot \mathrm{erfc}(-iz), \quad y^{(n)} = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n y(z). \tag{15}$$

The derivative with respect to ξ in (12) is transformed to a derivative with respect to z. Then the g_n are given by

$$g_n = (i/\gamma)\,\delta_{n0} + \delta_{n1} + \frac{\xi}{\gamma}\sqrt{\pi/2}(-1)^{n+1}\frac{1}{2^{n/2}} y^{(n)}(\xi/\sqrt{2}). \tag{16}$$

We differentiate (15) with respect to z, utilize the definition (14) and obtain directly and by induction

$$y^{(0)} = y(z), \quad y^{(1)} = -2z\,y(z) + 2i/\sqrt{\pi},$$
$$y^{(n+2)}(z) + 2z\,y^{(n+1)}(z) + 2(n+1)\,y^{(n)}(z) = 0,$$
$$n = 0, 1, 2, \ldots . \tag{17}$$

Eventually f_1 is given in the form of (7) with coefficients g_n

$$g_0 = C\cdot y^{(1)}(\xi/\sqrt{2}), \quad g_1 = 1 + C\cdot\xi\cdot y^{(1)}(\xi/\sqrt{2}),$$
$$g_n = C\cdot(-1)^{n+1}\, 2^{-(n-1)/2}\cdot\xi\cdot y^{(n)}(\xi/\sqrt{2}),$$
$$\text{for} \quad n \geq 2 \tag{18}$$

with $C = \sqrt{\pi}/2\gamma$ and $\xi = i/\gamma - u(x)$.

From (7) it is obvious that f_1 tends to zero if γ approaches zero, i.e. in the collision-dominated limit, and approaches $f_1 = f_0 \frac{\partial u}{\partial x} i\cdot(v-u)$ for $\gamma \to \infty$, i.e. in the collisionless limit. It is easily seen that in the collision-dominated limit with $\gamma \to 0$ the absolute value of ξ tends to infinity, $|\xi| \to \infty$, and the result

$$\lim_{\gamma \to 0} |g_n|/n! = 0 \tag{19a}$$

holds for all values of n.

In the collisionless limit with $\gamma \to \infty$ but $u \cdot \gamma$ bounded the absolute value of ξ tends to zero, $|\xi| \to 0$. One then has

$$\lim_{\gamma \to \infty} |g_0| = 1/\gamma = 0, \; \lim_{\gamma \to \infty} |g_1| = 1, \; \lim_{\gamma \to \infty} |g_n|/n! = 0. \tag{19b}$$

Consistent with the convergence of the series expansion (7) is the result for the asymptotic behaviour for a fixed γ, i.e. for bounded $|\xi|$,

$$\lim_{n \to \infty} g_n/n! \to 0. \tag{19c}$$

ii) Moment Series Expansion

In most problems only approximate solutions with a truncated series expansion are possible. The method consists in deriving first an infinite set of equations for the coefficients g_n, which is then approximated by a finite set. This set is obtained by truncation, i.e. by putting all $g_n = 0$ for $n > N$, where N is typically 3. In order to investigate the accuracy of this method we begin again with (6), which is put into the form

$$\left(v + \frac{1}{i\gamma}\right) f_1 = i v w(x) u'(x) f_0, \tag{20}$$

$$\left(v + \frac{1}{i\gamma}\right) f_1 = i u'(x) (\mathrm{He}_2(w) + u \, \mathrm{He}_1(w) + \mathrm{He}_0(w)) f_0,$$

when the explicit form of the first three Hermite polynomials is inserted. We multiply this equation by $\mathrm{He}_n(w)$ and integrate over v:

$$\int_{-\infty}^{+\infty} dv \, \mathrm{He}_n(w) \cdot (\xi - w) f_1 = -i u'(x) (\delta_{n0} + u(x) \delta_{n1} + 2\delta_{n2}). \tag{21}$$

Here the relation $w \cdot \mathrm{He}_n(w) = \mathrm{He}_{n+1} + n \mathrm{He}_{n-1}$ is applied. Inserting the series expansion (7) for f_1 we obtain the system

$$n \cdot g_{n-1} - \xi \cdot g_n + g_{n+1} = \delta_{n0} + u \delta_{n1} + 2\delta_{n2}, \quad n = 0, 1, 2, 3, \ldots \tag{22}$$

with $g_{-1} = 0$.

It is easily verified that the exact solution given by (18) satisfies the recursion (22). We truncate at $n = N$ obtaining $N + 1$ equations for the unknowns $g_0, g_1, \ldots, g_N$. The approximate solution is denoted by $\bar{g}_n$. The system of $N + 1$ equations reads

$$\begin{aligned} -\xi \cdot \bar{g}_0 + \bar{g}_1 &= 1, \\ \bar{g}_0 - \xi \cdot \bar{g}_1 + \bar{g}_2 &= u, \\ 2 \cdot \bar{g}_1 - \xi \cdot \bar{g}_2 + \bar{g}_3 &= 2, \\ 3 \cdot \bar{g}_2 - \xi \cdot \bar{g}_3 + \bar{g}_4 &= 0, \\ &\ldots \\ N \cdot \bar{g}_{N-1} - \xi \cdot \bar{g}_N &= 0. \end{aligned} \tag{23}$$

Let D_N be the determinant of the homogeneous system and D_i the determinant of the system, where the i-th column is replaced by the right-hand side vector. The solution $\bar{g}_i$ is then given by

$$\bar{g}_i = D_i / D_N. \tag{24}$$

We expand the determinant D_N in the elements of the last row,

$$D_N = -\xi \cdot D_{N-1} - N \cdot D_{N-2},$$

and obtain

$$D_N = \mathrm{He}_{N+1}(-\xi) = (-1)^{N+1} \mathrm{He}_{N+1}(\xi). \tag{25}$$

The solution for truncation at $N = 0$, 1, 2 and 3 is easily obtained and reads

$$\bar{g}_0 = -1/\xi \quad \text{for} \quad N = 0, \tag{26a}$$

$$\bar{g}_0 = (i/\gamma) \frac{1}{1 - \xi^2}, \; \bar{g}_1 = \frac{1 + u \cdot \xi}{1 - \xi^2}, \quad \text{for} \quad N = 1 \tag{26b}$$

$$\bar{g}_0 = (i/\gamma) \frac{1}{3 - \xi^2}, \; \bar{g}_1 = \frac{3 + u \cdot \xi}{3 - \xi^2}, \tag{26c}$$

$$\bar{g}_2 = (2i/\gamma) \frac{1}{3 - \xi^2}, \quad \text{for} \quad N = 2,$$

$$\bar{g}_0 = (i/\gamma) \frac{3 - \xi^2}{\xi^4 - 6\xi^2 + 3}, \quad \bar{g}_1 = 1 + \xi \cdot \bar{g}_0, \tag{26d}$$

$$\bar{g}_2 = \frac{2\xi^2 \cdot \bar{g}_0}{\xi^2 - 3}, \; \bar{g}_3 = \frac{6\xi \cdot \bar{g}_0}{\xi^2 - 3}, \quad \text{for} \quad N = 3.$$

We now examine the error due to the truncation. The difference

$$e_n = g_n - \bar{g}_n, \quad \text{for} \quad n \le N \tag{27}$$

again satisfies the recursion (22) with zeros on the right-hand side except for the last element being $-g_{N+1}$. The error e_n is determined again by (24) when D_i is replaced by the corresponding new determinant $\bar{D}_i$. The error due to truncations is now derived for the first element $\bar{g}_0$. The determinant $\bar{D}_0$ is evaluated by expanding with respect to the elements of the first column, which yields $(-1)^{N+2} \cdot (-g_{N+1}) \cdot \tilde{D}_0$. The determinant $\tilde{D}_0$ is given by the corresponding matrix where the first row and column are eliminated. This matrix now has in the first row a unit element, otherwise zeros. Consequently this determinant has unit value and we obtain the result

$$e_0 = \bar{D}_0/D_N = (-1)^{N+1}\, g_{N+1}/D_N = g_{N+1}/\mathrm{He}_{N+1}(\xi). \tag{28}$$

The error for the higher terms $\bar{g}_1, \bar{g}_2, \ldots$ can then easily be derived from the system of equations for the e_n

$$e_1 = \xi \cdot e_0,\; e_2 = (\xi^2 - 1) \cdot e_0,\; e_3 = (\xi^3 - 3\,\xi) \cdot e_0. \tag{29}$$

If truncation occurs for N with $N < n$ the error is obviously given by

$$e_n = g_n, \quad \text{for} \quad N < n. \tag{27b}$$

For an overview we again consider various limits.

α) $\gamma \to 0$

In the collision-dominated limit with $\gamma \to 0$ the absolute value of ξ tends to infinity, $|\xi| \to \infty$, and one has $g_{N+1} \propto \xi^{-N}$ and $\mathrm{He}_{N+1} \propto \xi^{N+1}$, which implies for e_0 the result $\lim_{\gamma \to 0} |e_0| = 0$. The same result holds for all the e_n with $n \le N$. This result is not surprising.

β) $\gamma \to \infty$

In the collisionless limit with $\gamma \to \infty$ but $u \cdot \gamma$ bounded the absolute value of ξ tends to zero, $|\xi| \to 0$. For the asymptotic behaviour of He_{N+1} the cases of even and odd N are distinguished. When N is even, i.e. $N = 2M$, then He_{N+1} has the limiting form $\mathrm{He}_{2M+1} = (N+1)!(-1)^M \xi/(M!\, 2^M)$ and hence $\lim_{\gamma \to \infty} |e_0| = 1/\gamma = 0$. When N is odd, i.e. $N = 2M - 1$, one then has $\mathrm{He}_{2M} = (-1)^M N!/(M!\, 2^M)$ and hence $\lim_{\gamma \to \infty} |e_0| = |\xi/\gamma| = 0$.

Let us give up now the condition that $|\gamma \cdot u|$ be bounded. If $\gamma \to \infty$ and u is fixed, it follows that $\xi \to -u$. With $z = -u/\sqrt{2}$ it is concluded from (15) that

$$y(z) = \mathrm{e}^{-u^2/2} \sqrt{2/\pi} \int_{iu/\sqrt{2}}^{+\infty} \mathrm{d}s\, \mathrm{e}^{-s^2} \tag{30}$$

when the relation (14) is inserted. Thus $y^{(n)}$ has a finite value and $g_{N+1} \propto (-u)/\gamma$. The error e_0 then becomes large at the zeros of $\mathrm{He}_{N+1}(-u)$. These zeros occur for $N = 1$ at $u_c = 1$, for $N = 2$ at $u_c = 1.73$, for $N = 3$ at $u_c = 0.74$ and 2.33 and for $N = 4$ at $u_c = 1.36$ and 2.86. The error e_0 stays finite and is asymptotically zero except for certain velocities close to u_c given by the zeros of $\mathrm{He}_{N+1}(-u)$. Thus we get the surprising result that except for these special velocities, which are of the order of the thermal velocity, the error approaches zero also in the collisionless limit. This might, however, be a special property of the model considered here.

γ) $N \to \infty$ with $|\xi|$ bounded

Owing to the convergence of the Hermite polynomial expansion we expect in this limit a vanishing error. To study the asymptotic behaviour for a fixed γ and large N the relation (13b) is inserted in (12) and we obtain for the coefficients g_n with $n > 1$

$$g_n = -(\xi/\gamma)(-i)^n \int_0^{+\infty} \mathrm{d}s\, s^n\, e^{-s^2/2 + i\xi s}. \tag{31}$$

With the relation

$$\sqrt{2\pi}\, e^{-\xi^2/2} = \int_{-\infty}^{+\infty} \mathrm{d}s\, e^{-s^2/2 + i\xi s}.$$

in the Rodriguez formula (9) the following representation of the Hermite polynomials is derived:

$$\mathrm{He}_n(\xi)\, e^{-\xi^2/2} = (-i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}s\, s^n\, e^{-s^2/2 + i\xi s}.$$

Then the error e_0, (28), is given in the form

$$e_0 = -(\xi/\gamma)\sqrt{2\pi}\, e^{-\xi^2/2} \frac{\int_0^{+\infty} \mathrm{d}s\, s^n\, e^{-s^2/2 + i\xi s}}{\int_{-\infty}^{+\infty} \mathrm{d}s\, s^n\, e^{-s^2/2 + i\xi s}}. \tag{32}$$

The integral in the denumerator is rewritten as

$$I = (-1)^n \int_0^{+\infty} \mathrm{d}s\, s^n\, e^{-s^2/2 - i\xi s} + \int_0^{+\infty} \mathrm{d}s\, s^n\, e^{-s^2/2 + i\xi s}, \tag{33}$$

where both integrands are of the form

$$\exp(n \ln s - s^2/2 \mp i\,\xi\, s). \tag{34}$$

For an approximate evaluation by use of the saddle-point method the exponent is expanded around its maximal value, which is given by

$$s_0 = \mp i\,\xi/2 + \sqrt{n - \xi^2/4}, \tag{35}$$

respectively. The second derivative of the exponent is $-n/s^2 - 1$ and has at $s = s_0$ the asymptotic value -2.

Therefore the value of these integrals is basically determined by the factor $\exp(n \ln s_0 - s_0^2/2 \mp i\xi s_0)$. The exponent is

$$E_0 = n \ln(\sqrt{n}\cdot(\sqrt{1-\xi^2/4n} \mp i\xi/2\sqrt{n})) - \xi^2/4 - n/2 \mp i\xi/2\sqrt{n-\xi^2/4}$$

which for large n becomes

$$E_0 \approx -n/2 + n \ln\sqrt{n} \mp i\xi\sqrt{n}. \tag{36}$$

Then we obtain for the error

$$e_0 = -(\xi/\gamma)\sqrt{2\pi}\, e^{-\xi^2/2} \frac{e^{i\xi\sqrt{n}}}{\mp e^{-i\xi\sqrt{n}} + e^{i\xi\sqrt{n}}}. \tag{37}$$

Since $\operatorname{Im}\xi = 1/\gamma > 0$ it holds with $n = N+1$ that

$$\lim_{N\to\infty} |e_0| \propto \frac{\xi}{\gamma} e^{-\xi^2/2}\, e^{-(2/\gamma)\sqrt{N+1}} \to 0. \tag{38}$$

Thus, the error e_0 approaches indeed zero for given $\gamma > 0$ and sufficiently large N, as one expects from the convergence of the Hermite expansion.

iii) Chapman-Enskog Scheme

In the Chapman-Enskog expansion the term $\mathbf{v}\cdot\nabla f_1 \propto ikvf_1$ is neglected in (6). Then f_1 is given by

$$f_1 = -v\gamma \frac{\mathrm{d}f_0}{\mathrm{d}x}, \tag{39}$$

whereas the exact solution is

$$f_1 = -\frac{v\gamma}{1+i\gamma v}\frac{\mathrm{d}f_0}{\mathrm{d}x}, \tag{39b}$$

It is easily seen that these expressions agree for small γ but differ for large γ. The solution (39) increases as γ but the exact solution becomes bounded.

For comparison, we cast f_1 into the form of (7) with the factor $(1+i\gamma)^{-1}v$ replaced by 1 and obtain

$$g_0 = i\gamma, \quad g_1 = iu\gamma, \quad \text{and} \quad g_2 = 2i\gamma. \tag{40}$$

3. Discussion

The physically relevant quantities of the system are the moments

$$\psi^\mu := \int_{-\infty}^{+\infty} \mathrm{d}v\, f\,(v-u(x))^\mu \quad \text{with} \quad \mu = 0, 1, 2, \text{and } 3. \tag{41}$$

The exact and approximative solutions derived in the previous section allow one to discuss explicitely the accuracy up to which these quantities are evaluated. The explicit form follows from (27) (or (27b), respectively), (28), (29), (18), and (23) together with (15), (17), and (26b). Let us begin with the density

$$n := \int_{-\infty}^{+\infty} \mathrm{d}v\, f = 1 + \int_{-\infty}^{+\infty} \mathrm{d}w\, f_1 = 1 + iu'(x)\, g_0. \tag{42}$$

Concerning this quantity it is appropriate to discuss the accuracy of $\tilde{n} = n-1$ with respect to truncation in the moment series expansion (7). The relative error is then given by

$$E(0)_N = \frac{g_0 - \bar{g}_0}{g_0} = e_0/g_0 = \frac{g_{N+1}}{\mathrm{He}_{N+1}(\xi)\cdot g_0}. \tag{43}$$

In the same fashion the Chapman-Enskog scheme yields the solution

$$n - 1 = -u'(x)\gamma \tag{44}$$

with a relative error

$$E(0)_{\mathrm{C.E.}} = \frac{g_0 - i\gamma}{g_0}. \tag{45}$$

For the first moments, the velocity, we get relative errors

$$E(1)_N = e_1/g_1 = 1.0 \quad \text{for} \quad N = 0, \tag{46a}$$

$$E(1)_N = e_1/g_1 = \frac{\xi\cdot e_0}{g_1} \quad \text{for} \quad N \ge 1, \tag{46b}$$

and

$$E(1)_{\mathrm{C.E.}} = \frac{g_1 - i\gamma u}{g_1}, \tag{47}$$

respectively.

For the pressure, $\mu = 2$, the result is

$$p = 1 + iu'\{g_0 + g_2\} \tag{48}$$

with a relative error in $\tilde{p} = p - 1$

$$E(2)_N = \frac{e_0 + g_2}{g_0 + g_2} \quad \text{for} \quad N = 0 \text{ and } 1, \tag{49a}$$

$$E(2)_N = \frac{\xi^2\cdot e_0}{g_0 + g_2} \quad \text{for} \quad N \ge 2, \tag{49b}$$

and

$$E(2)_{\mathrm{C.E.}} = 1 - \frac{3i\gamma}{g_0 + g_2}, \tag{50}$$

The last moment considered is the heat flux q, $\mu = 3$. We obtain

$$q = + i u' \{g_3 + 3 g_1\}, \tag{51}$$

and the relative error is given by

$$E(3)_N = 1.0 \quad \text{for} \quad N = 0, \tag{52a}$$

$$E(3)_N = \frac{3 e_1 + g_3}{3 g_1 + g_3} \quad \text{for} \quad N = 1 \text{ and } 2, \tag{52b}$$

$$E(3)_N = \frac{\xi^3 \cdot e_0}{3 g_1 + g_3} \quad \text{for} \quad N \geq 3, \tag{52c}$$

and

$$E(3)_{\text{C.E.}} = 1 - \frac{3 i \gamma u}{3 g_1 + g_3}. \tag{53}$$

Table 1. The relative error in the physically relevant quantities as a function of truncation; $\xi = i/\gamma - u(x)$, $e_0 = \dfrac{g_{n+1}}{\mathrm{He}_{N+1}(\xi)}$.

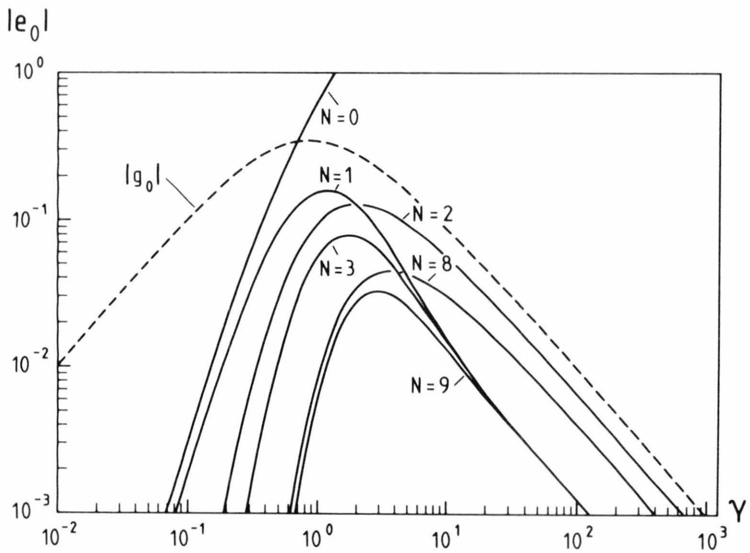
N	$n-1$	w	$p-1$	q
0	$\frac{e_0}{g_0}$	1	$\frac{g_2+e_0}{g_2+g_0}$	1
1	$\frac{e_0}{g_0}$	$\frac{e_1}{g_1} = \frac{\xi \cdot e_0}{g_1}$	‖	$\frac{g_3+3e_1}{g_3+3g_1}$
2	$\frac{e_0}{g_0}$	$\frac{e_1}{g_1}$	$\frac{e_0+e_2}{g_0+g_2} = \frac{\xi^2 \cdot e_0}{g_0+g_2}$	‖
3	$\frac{e_0}{g_0}$	$\frac{e_1}{g_1}$	$\frac{e_0+e_2}{g_0+g_2}$	$\frac{3e_1+e_3}{3g_1+g_3} = \frac{\xi^3 \cdot e_0}{3g_1+g_3}$

In Table 1 we have collected the formulae of the relative error of the physical moments when N terms are kept in the expansion. Clearly, higher moments such as the heat flux require sufficiently high truncation.

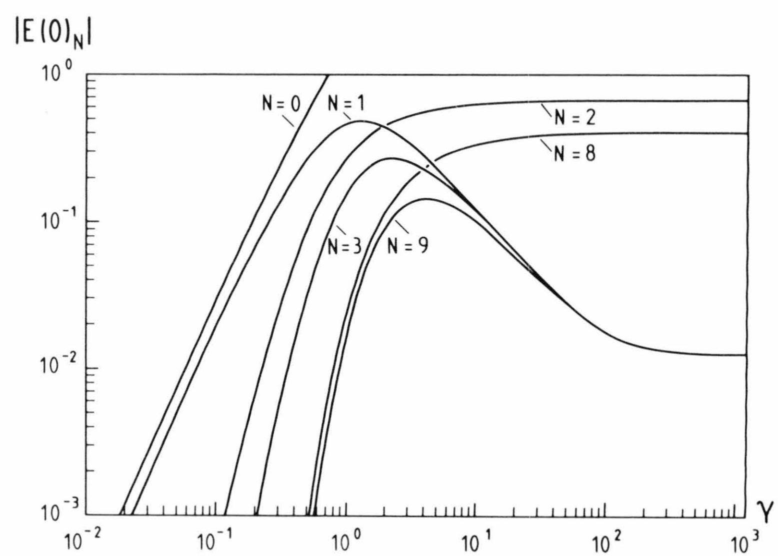
In Figs. 1–4 the errors of the physically relevant quantities are displayed in dependence of the collision time. The error in the density $\tilde{n} = n - 1$, is shown in Fig. 1. The absolute value of the error e_0 defined in (27) and (28), is displayed in Fig. 1 a) for a mean velocity $u = 0.1$. If only one moment, namely $N = 0$, is kept in (7) and (28), the error vanishes for small γ as $1/\xi$ and increases with γ for large γ. This is immediately evident from the asymptotic limits of (28); it is recalled that $\mathrm{He}_1(\xi) = \xi$. For $N \geq 1$ the error vanishes both for small and for large γ in agreement with the asymptotic values of the g_n discussed above. Clearly, the error decreases with increasing number of expansion terms. The behaviour for even and odd values of N is different. The dependence of the expansion coefficient g_0 with respect to γ is also displayed in Figure 1 a. It is emphasized that in the limit $\gamma \to \infty$ the term g_0 decreases as $1/\gamma$. Similar results hold for different values of the mean velocity u.

Now the relative errors are considered. The absolute value of the relative error in the density $\tilde{n} = n - 1$, given by (42) and (43), is shown in Fig. 1 b) for a mean velocity $u = 0.01$. If only one moment, namely $N = 0$, is kept in (7) and (43), the error vanishes for small γ as $1/\xi$ and increases with γ for large γ. This is again immediately evident from the asymptotic limits of (43). For $N \geq 1$ the error is bounded and decreases for large γ, which is again easily seen from the limit $\gamma \to \infty$ of (43). Since the quantity g_0 and its error e_0 both vanish for large γ as $1/\gamma$, the relative error $E(0)_N$ assumes then a constant value. The behaviour for even and odd values of N is different, but the error decreases as the number of moments increases. It is emphasized that for $N = 9$ the error is less than 10% and further that for $N = 3$, i.e. with the heat flux included, the error is less than 30% and rapidly decreases for large γ. If the mean flow velocity $u(x)$ is increased, the error increases accordingly for large γ, especially if $\gamma \cdot u$ is larger than unity. The case for $u = 1.0$ is plotted in Fig. 1 c. Here the error increases with γ for $N = 0$ and $N = 1$. The Hermite polynomial $\mathrm{He}_2(\xi) = \xi^2 - 1$ scales for large γ and $u = 1.0$ as $\mathrm{He}_2(\xi) \approx -2i \cdot u/\gamma \to 0$ and, hence, the error diverges in this case. For $N \geq 2$ the error remains bounded. In the collision-dominated regime the error decreases when the number of Hermite polynomials is sufficiently large. In the collisionless regime the relative error is finite but not small. It is emphasized that the case where the mean flow velocity equals the thermal speed is mostly not of physical relevance. Nevertheless, it is interesting that this method still produces reasonable results.

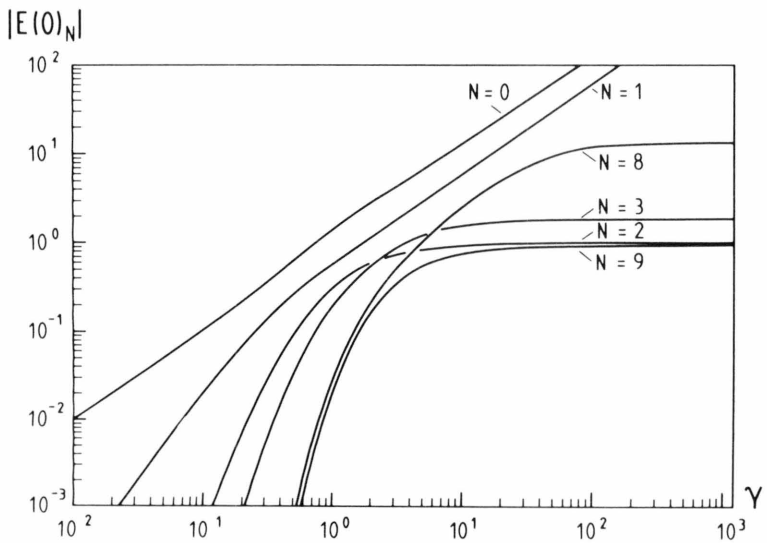
The error for the velocity ($\mu = 1$ in (41)) given by (46 a) and (46 b) is shown in Figure 2. For $N = 0$ the error is unity by definition and is large for $N = 1$ in the collision-dominated regime. This is due to the fact that $e_1 = \xi \cdot e_0$ and that these ξ factors cancel. In Fig. 2 a the mean flow velocity is small, $u = 0.01$. It is seen that for $N \geq 2$ the error is small and vanishes in both limits $\gamma \to 0$ and $\gamma \to \infty$. The case of $u = 1.0$, Fig. 2 b, yields for $N = 1$ a large error for small as well as for large γ owing to the behaviour of $\mathrm{He}_2(\xi)$ as discussed above. For truncation at $N = 3$ the error is quite small every-



a



b



c

Fig. 1a. The absolute value of the error of the zeroth moment $e_0 = g_0 - \bar{g}_0$ defined in (28), where g_0 is the exact solution and $\bar{g}_0$ is obtained by truncation at $n = N$, as a function of collision time for a mean volocity $u = 0.1$. In normalized units, $\gamma = \tau\, v_{th}\, k$ denotes the ratio of the mean-free path and the scale length k^{-1}. The number of truncation N is indicated on the curves. For comparison the value of g_0 (dotted line) is given.
Fig. 1b. The absolute value of the relative error for the density $\tilde{n} = n - 1$ defined in (43) as a function of collision time for a mean velocity $u = 0.01$. In normalized units $\gamma = \tau\, v_{th}\, k$ denotes the ratio of the mean-free path and the scale length k^{-1}. The number of truncation N is indicated on the curves.
Fig. 1c. The absolute value of the relative error for the density $\tilde{n} = n - 1$ defined in (43) as a function of collision time for a mean velocity $u = 1.0$. In normalized units $\gamma = \tau\, v_{th}\, k$ denotes the ratio of the mean-free path and the scale length k^{-1}. The number of truncation N is indicated on the curves.

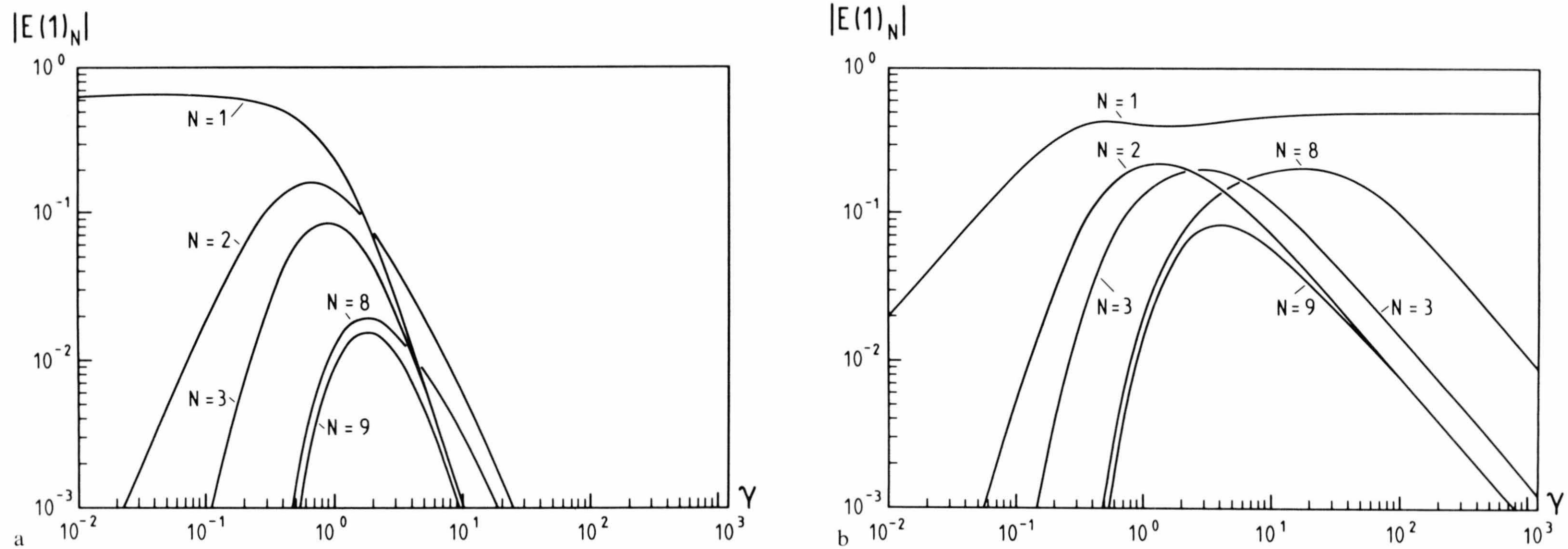


Fig. 2. The absolute value of the relative error for the velocity w defined in (46 a) and (46 b) as a function of collision time. In normalized units $\gamma = \tau\, v_{\text{th}}\, k$ denotes the ratio of the mean-free path and the scale length k^{-1}. The number of truncation N is indicated on the curves. a) for a mean velocity $u = 0.01$. b) for a mean velocity $u = 1.0$.

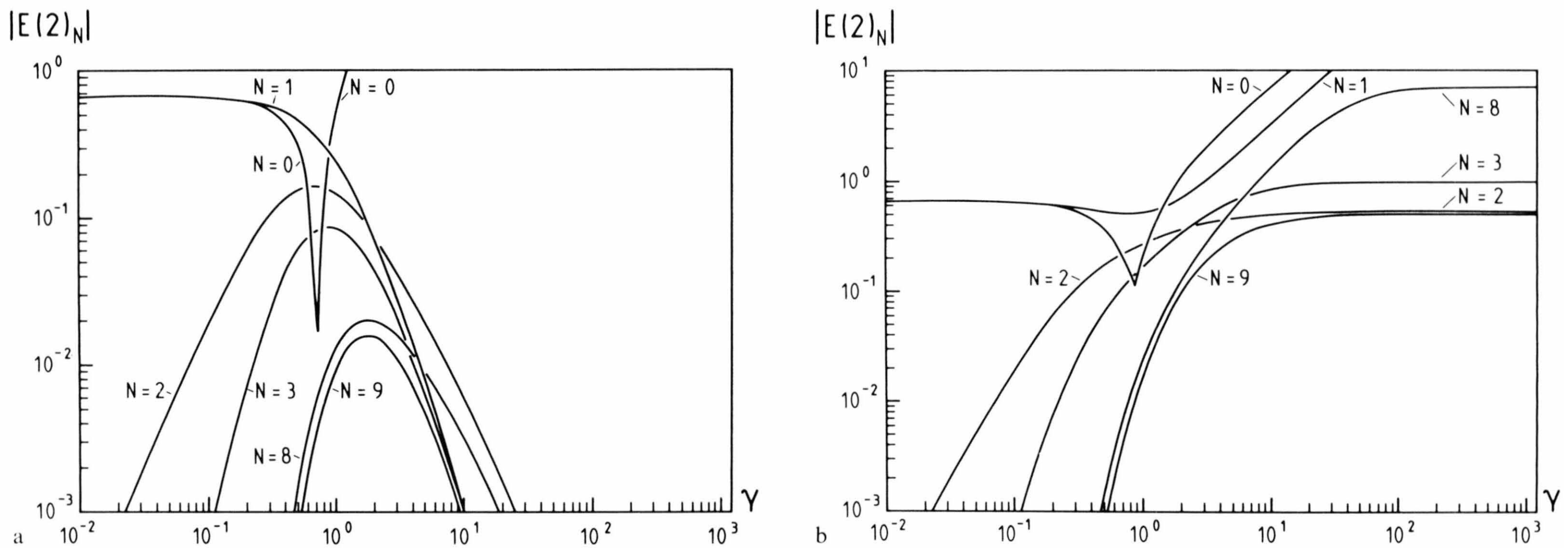


Fig. 3. The absolute value of the relative error for the pressure $\tilde{p} = p - 1$ defined in (48) and (49 a, b) as a function of collision time. In normalized units $\gamma = \tau\, v_{\text{th}}\, k$ denotes the ratio of the mean-free path and the scale length k^{-1}. The number of truncation N is indicated on the curves. a) for a mean velocity $u = 0.01$. b) for a mean velocity $u = 1.0$.

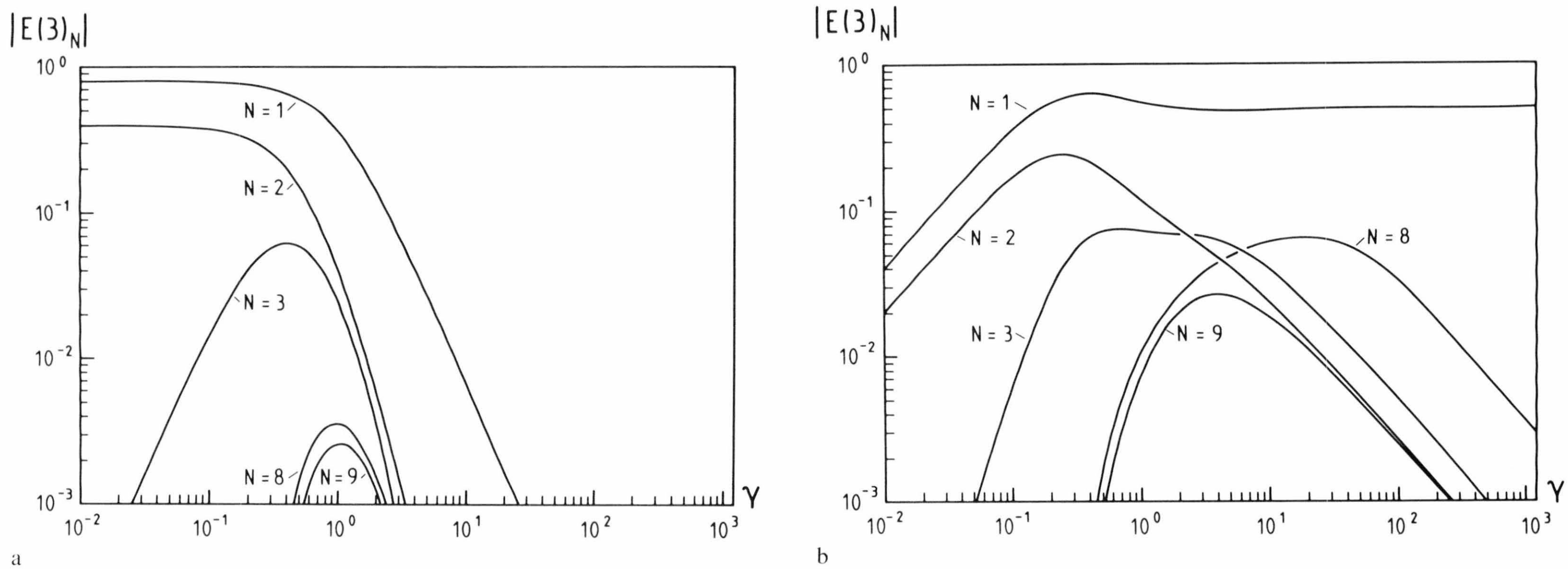


Fig. 4. The absolute value of the relative error for the heat flux q defined in (51) and (52a–c) as a function of collision time. In normalized units $\gamma = \tau\, v_{\mathrm{th}}\, k$ denotes the ratio of the mean-free path and the scale length k^{-1}. The number of truncation N is indicated on the curves. a) for a mean velocity $u = 0.01$. b) for a mean velocity $u = 1.0$.

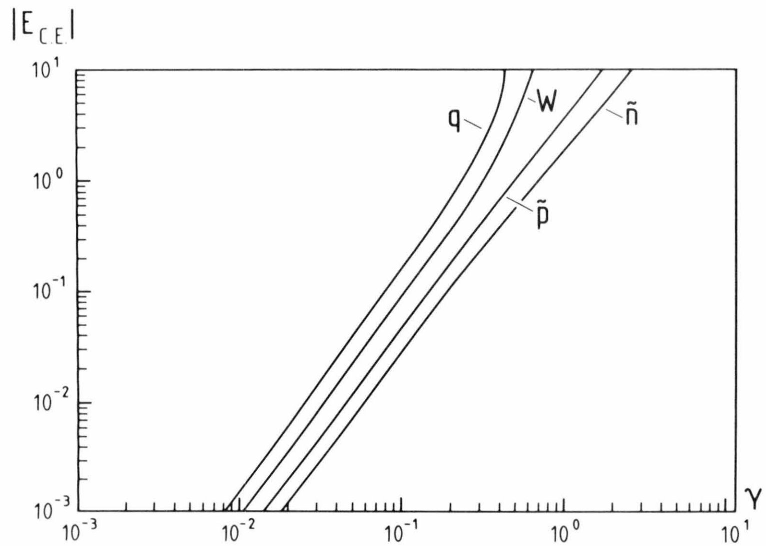


Fig. 5. The absolute value of the relative error as a function of collision time of the Chapman-Enskog expansion for the density $\tilde{n} = n - 1$, velocity w, pressure $\tilde{p} = p - 1$, and heat flux q defined in (45), (47), (50) and (53). In normalized units $\gamma = \tau\, v_{\mathrm{th}}\, k$ denotes the ratio of the mean-free path and the scale length k^{-1}.

where. Especially in the collision-dominated regime the error decreases with sufficiently high N. It is recalled that g_1 approaches unity for large γ.

The pressure $\tilde{p}(\mu = 2$ in (41)) is approximated well only for $N \geq 2$. But let us, nevertheless, discuss the relative error also for $N = 0, 1$. In Fig. 3a the error is displayed as a function of γ for $u = 0.01$. For $N = 0$ and 1 the error is given according to (27b) and (49a) by

$$E(2)_N = \frac{g_2 + g_0 - \bar{g}_0}{g_0 + g_2} = 1 - \frac{\bar{g}_0}{g_0 + g_2}. \tag{54}$$

For $N \geq 2$ (49b) applies. In the collision-dominated regime, i.e. $\gamma \to 0$, g_0 as well as g_2 are proportional to γ as follows from (18) and (19). From (26a, b) the result for $\bar{g}_0$ is known and, when inserted into (54), the γ factors cancel and the error assumes as constant value. The collisionless limit, $\gamma \to \infty$ with $|\xi| \to -u$ for fixed velocity, yields a different behaviour. Equation (23) is used to express $g_0 + g_2$ by $u + \xi \cdot g_1$ with $\lim_{\gamma \to \infty} g_1 = 1$, see (19b). It follows from (18), (19), and (26a) that the error diverges as γ for $N = 0$. For $N = 1$ the error is given by

$$E(2)_N \approx 1 - \frac{1}{1 - \xi^2} = -\frac{\xi^2}{1 - \xi^2} \to 0 \tag{54b}$$

and vanishes except in the case $\xi^2 \approx 1$. Thus for $u = 1.0$, where $\xi^2 - 1 \approx -2i \cdot u/\gamma$, the error again diverges for large γ as is seen from Figure 3b). For truncation at $N \geq 2$ in the Hermite polynomial expansion the error remains bounded as is seen from Figs. 3a and b. It is emphasized that for $N = 3$ the relative error is less than 10% and rapidly vanishes in both limits $\gamma \to 0, \infty$ for small mean flow velocity u.

The last quantity discussed is the heat flux ($\mu = 3$ in (41). For $N = 0$ the error is by definition unity and is given for $N = 1$ and 2 according to (27b) and (52b) by

$$E(3)_N = 1 - \frac{3\bar{g}_1}{g_3 + 3g_1}. \tag{55}$$

For $N \geq 3$ the error is given by (52c). The error is displayed in Fig. 4 for $u = 0.01$ and $u = 1.0$. In the collision-dominated regime the error decreases with $\gamma \to 0$. This follows from the limiting forms of the above formulae and is clearly seen in the figures for $u = 1.0$, while for $u = 0.01$ the error is close to unity at $\gamma = 10^{-2}$. In the collisionless limit the denumerator assumes the value $3g_1 \approx 3$. The result in (26b) shows that $\bar{g}_1 \to 1$ for $N = 1, 2$ and, thus, the error vanishes. In the case of $u = 1.0$ the denumerator $1 - \xi^2$, (26b), causes the different asymptotic behaviour. The error for higher truncation remains bounded and is quite small. For $N = 3$ the error is less than 10% everywhere. These findings agree with the asymptotic limits derived above.

In summary, it has been explicitly shown that the moment method based on a Hermite series expansion including density, velocity, pressure and heat flux, i.e. for $N = 3$, yields sufficiently accurate results for all values of the collision time if the flow velocity is sufficiently small ($\gamma \cdot u(x) < 1$). In particular, the mean-free path need not be smaller than the connection length. The validity of the fluid model is guaranteed if the distribution function is close to a Maxwellian. The scheme yields reasonable results even in the case of large flow where the flow velocity becomes comparable to the thermal speed. For tokamak transport the plasma flow is significantly smaller. Following the Chapman-Enskog scheme, neglecting the term $i\gamma v f_1$ in (4) and (6) yields a wrong dependence of the distribution function f_1 on γ for large γ. Whereas the exact solution f_1 is bounded, this solution increases as γ. The relative error of the various moments with respect to the exact solution is displayed in Figure 5. Obviously, the accuracy of this approximation is only good for $\gamma < 1$. The dependence on the velocity u is then, of course, not pronounced.

This means that the Chapman-Enskog scheme is only accurate in the collision-dominated regime.

4. Conclusions

In this paper we have demonstrated with an example that a Hermite series expansion of a kinetic equation gives a fair approximation to the original kinetic equation. Such moment series were used to model transport many years ago in two fundamental papers [12], [13]. Both of these papers developed a system of equations for total momentum and total energy transfer as well as an Ohm's law and equations for each species for stress and heat flow. Braginskii [2] used a Chapman-Enskog expansion to obtain a system of fluid equations for each species. The justifications for a Chapman-Enskog expansion and for a Hermite series expansion are drastically different, the range of validity of the second approach extending far beyond that of the first method, as is evident from our model equation.

The basic conclusion of our study is the validity of a certain macroscopic modelling of tokamaks in the low-collisionality regime at least as far as particle trapping and untrapping does not play a role. This problem will be investigated in a following paper on the basis of a simple Fokker-Planck type model equation.

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